ON A PARTICULAR CLASS OF SIGMA-MONOGENIC FUNCTIONS

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "On a Particular Class of Sigma-monogenic Functions", submitted by RALPH JORSTAD in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

This thesis is devoted to a study of a particular class of sigma-monogenic functions, namely those whose real and imaginary parts, u and v, satisfy

$$e^{-cx} u_x = v_y$$
$$e^{-cx} u_y = -v_x \quad .$$

The idea of a sigma-monogenic function is explained in Chapter I, and by solving Cauchy problems we show how to express our Σ-monogenic functions in terms of analytic functions. The concept of correspondence between Σ-monogenic and analytic functions is also defined here. In Chapter II, the formal powers $Z^{(n)}(z_o;z)$ are given and bounds on $|a \cdot Z^{(n)}(z_o;z)|$ are determined. Then in Chapter III, bounds on $f^{[n]}(z)$, the derivatives of a Σ-monogenic function, analogous to Cauchy's inequality, are derived by solving a Dirichlet problem in a circle. We are thus able to show that $f(z) = \sum_{0}^{\infty} \frac{f^{[n]}(z_o)}{n!} Z^{(n)}(z_o;z)$ for all $|z-z_o| < a$ provided $f(z)$ is Σ-monogenic in that region.

In the Appendix, an alternate proof is given for our formulae expressing a Σ-monogenic function in terms of an analytic function.

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CHAPTER I

INTRODUCTION

In 1944, Bers and Gelbart [1] considered the system of equations

$$\begin{aligned} \sigma_1(x)\, u_x &= \tau_1(y)\, v_y \\ \sigma_2(x)\, u_y &= -\tau_2(y)\, v_x \end{aligned} \tag{1.1}$$

(where the subscripts denote partial derivatives). They let $z = x + iy$, $f(z) = u + iv$ and introduced the Σ-derivative

$$\frac{d_\Sigma f}{d_\Sigma z} = f^{[1]}(z) = \sigma_1 u_x + i\,\frac{v_x}{\sigma_2} = \tau_1 v_y - i\,\frac{u_y}{\tau_2} \tag{1.2}$$

and the Σ-integral

$$\int_{z_o}^{z} f(z)\, d_\Sigma z = \int_{z_o}^{z} \sigma_2\, u\, dx - \tau_2\, v\, dy + i\left(\int_{z_o}^{z} \frac{1}{\sigma_1} v\, dx + \frac{1}{\tau_1} u\, dy\right). \tag{1.3}$$

Functions $f(z)$, where u and v are of class C^2 satisfying (1.1) were called sigma-monogenic (Σ-monogenic), where

$$\Sigma = \begin{pmatrix} \sigma_1 & \tau_1 \\ \sigma_2 & \tau_2 \end{pmatrix}.$$

Then $f^{[1]}(z)$ and $\int_{z_o}^{z} f(z)\, d_\Sigma z$ are Σ'-monogenic where

$$\Sigma' = \begin{pmatrix} \frac{1}{\sigma_2} & \tau_1 \\ \frac{1}{\sigma_1} & \tau_2 \end{pmatrix}.$$

Clearly $\Sigma'' = \Sigma$.

They then introduced the generalized powers $a \cdot Z^{(n)}(z_0;z)$, where a is a complex constant, by the equations ($\sim$ denotes class Σ'):

$$
\begin{aligned}
a \cdot Z^{(0)}(z_0;z) &= a \cdot \tilde{Z}^{(0)}(z_0;z) = a \\
a \cdot Z^{(1)}(z_0;z) &= \int_{z_0}^{z} a \, d_{\Sigma'} \zeta \\
a \cdot \tilde{Z}^{(1)}(z_0;z) &= \int_{z_0}^{z} a \, d_{\Sigma} \zeta \qquad (1.4) \\
a \cdot Z^{(n)}(z_0;z) &= n \int_{z_0}^{z} a \cdot \tilde{Z}^{(n-1)}(z_0;\zeta) d_{\Sigma'} \zeta \\
a \cdot \tilde{Z}^{(n)}(z_0;z) &= n \int_{z_0}^{z} a \cdot Z^{(n-1)}(z_0;\zeta) \, d_{\Sigma} \zeta \, .
\end{aligned}
$$

Bers and Gelbart showed that if $f(z)$ is Σ-monogenic in a circle of radius R about z_0, (σ_i, τ_i positive analytic) then

$$f(z) = \sum_{0}^{\infty} a_n \cdot Z^{(n)}(z_0;z) \qquad (1.5)$$

for $|z-z_0|$ sufficiently small. The a_n are uniquely determined by a "Taylor formula" as

$$a_n = \frac{f^{[n]}(z_0)}{n!}$$

where $f^{[n]}(z)$ is the n-th Σ-derivative of $f(z)$.

In later work [2], σ_i and τ_i were assumed to be positive and of class C^2 whereas in [1], σ_i and τ_i were analytic functions of their respective real variables.

One of the main unsolved problems in the theory of Σ-monogenic functions is:

If $f(z)$ is Σ-monogenic in $\Gamma = \{z;\ |z-z_o| < R\}$, is

$$f(z) = \sum_{0}^{\infty} \frac{f^{[n]}(z_o)}{n!} \cdot Z^{(n)}(z_o;z) \quad \text{in} \quad \Gamma \text{ ?}$$

We have succeeded in giving an affirmative answer to this question in the special case where $\sigma_1 = \sigma_2 = e^{-cx}$, $\tau_1 = \tau_2 = 1$.

One important procedure is to set up a correspondence between analytic functions and Σ-monogenic functions; that is, given any analytic function $f(z)$, we can find explicitly a Σ-monogenic function (for our special case) which has some of the properties of $f(z)$. This is done by solving Cauchy problems on a parallel to the y-axis.

We consider then the special case

$$\begin{aligned} e^{-cx} u_x &= v_y \\ e^{-cx} u_y &= -v_x \end{aligned} \qquad (1.6)$$

and, for convenience, assume that $c > 0$.

Assuming that u,v are of class C^2, it easily follows that

$$\Delta u - cu_x = u_{xx} + u_{yy} - cu_x = 0$$

$$\text{and} \quad \Delta v + cv_x = 0 \ .$$

The former equation is that satisfied by the vorticity in plane Oseen flow [3].

It is easy to see that the general equation with constant coefficients

$$\Delta u + a\, u_x + b\, u_y + c\, u = 0$$

can be reduced to this form provided $a^2 + b^2 - 4c \geq 0$. This is the case when the solution of the Dirichlet problem for the equation is unique [Paraf's Theorem].

In considering the [redacted] problem[redacted] mentioned above we shall assume $z_o = 0$. This is done merely to simplify the work: the general case can be handled in the same manner. Also the general formulae can be easily obtained from ours by a simple transformation.

For, let $z_o = x_o + i\, y_o$ and suppose we are interested in equations (1.6) in a neighbourhood of z_o. Let $X = x-x_o$, $Y = y-y_o$, $U = e^{-cx_o}u$, $V = v$. Then

$$e^{-cX}\, U_X = V_Y$$

$$e^{-cX}\, U_Y = -V_X$$

and so we can consider the last equation in a neighbourhood of $X = Y = 0$.

We now give the solutions of some Cauchy problems.

THEOREM 1.1: Suppose $f(z)$ is holomorphic in a circle with centre the origin and radius R . Let D be the square whose vertices are $(-R,0)$, $(0,R)$, $(R,0)$, and $(0,-R)$.

i) Then, inside D , the function

$$u(x,y) = f(y) + \frac{x}{\pi}\int_0^1 e^{cux}(1-u)^{1/2}\, u^{-1/2}\, du \int_0^\pi i \cos\phi\; f'(y+2ix[u(1-u)]^{1/2}\cos\phi)\, d\phi$$

is the solution of $\Delta u - cu_x = 0$ satisfying the conditions

$[u = f(y)\ ,\ \frac{\partial u}{\partial x} = 0\ ]$ on $x = 0$.

ii) $$u(x,y) = \frac{x}{\pi}\int_0^1 e^{cux}\, du \int_0^\pi f(y + 2ix[u(1-u)]^{1/2}\cos\phi)\, d\phi$$

is the solution in D of $\Delta u - c\,u_x = 0$, satisfying

$[u = 0\ ,\ \frac{\partial u}{\partial x} = f(y)]$ on $x = 0$.

iii) Inside D , the function

$$v(x,y) = f(y) + \frac{x}{\pi}\int_0^1 e^{-cux}(1-u)^{1/2}\, u^{-1/2}\, du \int_0^\pi i \cos\phi\; f'(y+2ix[u(1-u)]^{1/2}\cos\phi)\, d\phi$$

is the solution of $\Delta v + cv_x = 0$ satisfying the conditions

$[v = f(y)\ ,\ \frac{\partial v}{\partial x} = 0]$ on $x = 0$.

iv) $$v(x,y) = \frac{x}{\pi}\int_0^1 e^{-cux}\, du \int_0^\pi f(y + 2ix\ [u(1-u)]^{1/2}\cos\phi)\, d\phi$$

is the solution in D of $\Delta v + cv_x = 0$, satisfying

$[v = 0,\ \frac{\partial v}{\partial x} = f(y)]$ on $x = 0$.

PROOF: We shall prove i) as the proofs of iii), iv) are the same as those for i), ii). ii) is proved in a similar fashion to i) and so we only point out the main differences.

i) First, it is clear that the u defined in i) above satisfies the Cauchy data.

To prove that it satisfies the equation, note that

$$f'(y + 2ix[u(1-u)]^{1/2} \cos \phi) = f'(y) + (2ix[u(1-u)]^{1/2} \cos \phi)\, f''(y) + \dots$$
$$+ \frac{(2ix[u(1-u)]^{1/2} \cos \phi)^n}{n!} f^{(n+1)}(y) + \dots$$

provided $(x,y) \in D$.

So, since $\int_0^{\pi} \cos^{2n+1} \phi \, d\phi = 0$, $\int_0^{\pi} \cos^{2n} \phi \, d\phi = \pi \frac{(2n)!}{2^{2n}(n!)^2}$,

we get

$$u = f(y) + \int_0^1 e^{cux} du \left[\sum_1^{\infty} (-1)^n (1-u)^n u^{n-1} \frac{x^{2n}}{n!(n-1)!} f^{(2n)}(y) \right]$$
$$= \sum_0^{\infty} (-1)^n a_{2n}(x) f^{(2n)}(y)$$

where $a_o = 1$,

$$a_{2n}(x) = \frac{x^{2n}}{n!(n-1)!} \int_0^1 e^{cux} (1-u)^n u^{n-1} du .$$

The term-by-term integration in the above is valid in D as can be seen by using Cauchy's inequality for $f^{(2n)}(y)$.

Note also that

$$|a_{2n}(x)| \leq \frac{|x|^{2n}}{(2n)!} e^{|cx|}$$

since $\displaystyle\int_0^1 (1-u)^n u^{n-1} du = \frac{n!(n-1)!}{(2n)!}$.

Now, if $D \equiv \dfrac{d}{dx}$,

$$(D^2 - cD)a_2 = 1 \quad , \quad (D^2 - cD)a_{2n+2} = a_{2n} .$$

To show this write

$$a_{2n}(x) = \frac{1}{n!(n-1)!} \int_0^x (x-t)^n t^{n-1} e^{ct} dt$$

and do an integration by parts after differentiating.

Finally then, differentiating

$$u = \sum_0^\infty (-1)^n a_{2n}(x) f^{(2n)}(y)$$

term-by-term it is easily seen that $\Delta u - cu_x = 0$.

ii) The proof of ii) proceeds as above. The only difference is that instead of a_{2n} we have

$$A_0 = x\int_0^1 e^{cux} du = \int_0^x e^{ct} dt \quad ,$$

$$A_{2n} = \frac{x^{2n+1}}{(n!)^2} \int_0^1 e^{cux} u^n(1-u)^n du$$

$$= \frac{1}{(n!)^2} \int_0^x e^{ct} t^n (x-t)^n dt .$$

Again, $(D^2 - cD)A_{2n} = A_{2n-2}$, $(n \geq 1)$. Q.E.D.

COROLLARY 1.1: If $f(z)$ is an entire function, then the u's and v's defined above are solutions in the whole plane. If $f(z)$ has a singularity at $z = 0$, then u and v are solutions for $|x| < |y|$.

We are now in a position to express our Σ-monogenic functions in terms of an analytic function.

THEOREM 1.2: i) Suppose $f(z)$ is holomorphic in a circle $|z| < R$. Then, in the square D defined in Theorem 1.1,

$$u_1 + iv_1 = f(y) + \frac{x}{\pi}\int_0^1 e^{cux}\,(1-u)^{1/2}\,u^{-1/2}\,du\int_0^\pi i\cos\phi\, f'(y+2ix[u(1-u)]^{1/2}\cos\phi)\,d\phi$$

$$- i\,\frac{x}{\pi}\int_0^1 e^{-cux}\,du\int_0^\pi f'(y+2ix[u(1-u)]^{1/2}\cos\phi)\,d\phi$$

is Σ-monogenic and satisfies the Cauchy conditions

$$u_1(x=0) = f(y)\ ,\quad \frac{\partial v_1(x=0)}{\partial x} = -f'(y)\ ,\quad \frac{\partial u_1(x=0)}{\partial x} = v_1(x=0) = 0\ .$$

ii) Under the above assumptions about domains,

$$u_2 + iv_2 = \frac{x}{\pi}\int_0^1 e^{cux}\,du\int_0^\pi g'(y+2ix[u(1-u)]^{1/2}\cos\phi)\,d\phi$$

$$+ i\left\{g(y) + \frac{x}{\pi}\int_0^1 e^{-cux}\,(1-u)^{1/2}\,u^{-1/2}du\int_0^\pi i\cos\phi\; g'(y+2ix\,\cdot\right.$$

$$\left.\cdot\,[u(1-u)]^{1/2}\cos\phi)\,d\phi\right\}$$

is Σ-monogenic and satisfies the Cauchy data

$$u_2(x=0) = \frac{\partial v_2(x=0)}{\partial x} = 0\ ,\quad v_2(x=0) = g(y)\ ,$$

$$\frac{\partial u_2(x=0)}{\partial x} = g'(y)\ .$$

PROOF: i) As in Theorem 1.1, inside D we have

$$u = \sum_{0}^{\infty} (-1)^n a_{2n}(x) f^{(2n)}(y)$$

$$v = \sum_{0}^{\infty} (-1)^{n+1} \bar{A}_{2n}(x) f^{(2n+1)}(y) ,$$

where $a_o(x) = 1$,

$$a_{2n}(x) = \frac{1}{n!(n-1)!} \int_0^x e^{ct} (x-t)^n t^{n-1} dt$$

and $\bar{A}_o = \int_0^x e^{-ct} dt$

$$\bar{A}_{2n-2} = \frac{1}{[(n-1)!]^2} \int_0^x e^{-ct} (x-t)^{n-1} t^{n-1} dt .$$

Now, we easily see that

$\bar{A}_{2n-2} - e^{-cx} a'_{2n} = 0 \quad (n \geq 1)$ where $'$ denotes differentiation with respect to x .

So,

$$e^{-cx} u_x = v_y .$$

Also $\bar{A}'_{2n} - e^{-cx} a_{2n} = 0 \quad (n \geq 0)$

and thus $e^{-cx} u_y = -v_x$.

ii) With u and v defined in ii) we have

$$u = \sum_{0}^{\infty} (-1)^n A_{2n}(x)\, g^{(2n+1)}(y)$$

$$v = \sum_{0}^{\infty} (-1)^n \bar{a}_{2n}(x)\, g^{(2n)}(y)$$

where $\bar{a}_o = 1$

$$\bar{a}_{2n} = \frac{1}{n!(n-1)!} \int_0^x e^{-ct} (x-t)^n t^{n-1}\, dt$$

and $A_o = \int_0^x e^{ct}\, dt$

$$A_{2n} = \frac{1}{(n!)^2} \int_0^x e^{ct} (x-t)^n t^n\, dt .$$

Now, $e^{-cx} A'_{2n} - \bar{a}_{2n} = 0 \qquad (n \geq 0)$,

and so $e^{-cx} u_x = v_y$.

Also, $e^{-cx} A_{2n-2} - \bar{a}'_{2n} = 0 \qquad (n \geq 1)$

and so $e^{-cx} u_y = -v_x$. Q.E.D.

We can now define a Σ-monogenic function "corresponding" to an analytic function.

DEFINITION 1.1: If $F(z)$ is an analytic function in $|z| < R$, then the Σ-monogenic function corresponding to $F(z)$ at $z_o = o$ is

$$U + iV = (u_1 + iv_1) + (u_2 + iv_2)$$

where $(u_1 + iv_1)$, $(u_2 + iv_2)$ are defined in Theorem 1.2, and where

$$f(y) = \operatorname{Re} F(0 + iy)$$

$$g(y) = \operatorname{Im} F(0 + iy) \ .$$

$U + iV$ is defined in the square D described in Theorem 1.1.

For example, let $F(z) = x + iy$. Then $f(y) = 0$, $g(y) = y$.

So

$$U + iV = 0 + \frac{x}{\pi}\int_0^1 e^{cux}\, du \int_0^\pi d\phi + i[y + \frac{x}{\pi}\int_0^1 e^{-cux}(1-u)^{1/2} u^{-1/2} \cdot du \int_0^\pi i \cos\phi \, d\phi \,]$$

$$= \frac{1}{c}(e^{cx}-1) + iy \ .$$

This is $1 \cdot Z^{(1)}(0;z) = Z^{(1)}(0;z)$.

Again, let $F(z) = iz = ix - y$. Then $f(y) = -y$, $g(y) = 0$,

and

$$U + iV = -y - \frac{x}{\pi}\int_0^1 e^{cux}(1-u)^{1/2} u^{-1/2} du \int_0^\pi i \cos\phi \, d\phi + \frac{x}{\pi}\int_0^1 e^{-cux} du \int_0^\pi d\phi$$

$$= -y + \frac{i}{c}(1-e^{-cx}) \ .$$

This is $i \cdot Z^{(1)}(0;z)$.

We shall show that these examples indicate the general situation, namely that the Σ-monogenic function corresponding to z^n is $Z^{(n)}(0;z)$ and the function corresponding to iz^n is $i \cdot Z^{(n)}(0;z)$. Thus the function corresponding to az^n is $a \cdot Z^{(n)}(0;z)$.

It therefore follows that if $F(z) = \sum_{0}^{\infty} a_n z^n$, $|z| < R$, then the corresponding function is $\sum_{0}^{\infty} a_n \cdot Z^{(n)}(0;z)$, $z \in D$. Thus our definition of a corresponding function coincides with that of Bers and Gelbart.

Unfortunately, given a Σ-monogenic function $U + iV$, we do not know how to find the analytic function to which it corresponds. Thus we can not prove the Generalized Taylor's Theorem in this manner.

CHAPTER II

THE POWERS $a\cdot Z^{(n)}(z_o;z)$ AND THEIR BOUNDS.

We first give explicit formulae for the powers $Z^{(n)}(z_o;z)$ and $i\cdot Z^{(n)}(z_o;z)$ defined originally by Bers and Gelbart by equations (1.4). Then using these we show that

$$|a\cdot Z^{(n)}(0;z)| < 2^{3/2}(n+1)\,|a|\,e^{|cx|}\,r^n, \quad (r = (x^2+y^2)^{1/2} = |z|).$$

We next show that if $f(z)$ is Σ- monogenic in a circle $|z| < R$, where

$$\Sigma = \begin{pmatrix} e^{-cx} & 1 \\ e^{-cx} & 1 \end{pmatrix},$$

then

$$|f^{[n]}(0)| < \frac{K\,n!}{r^n}$$

where K does not depend on n and $r < R$. These bounds then suffice to show that $\sum_0^\infty \frac{f^{[n]}(o)}{n!}\cdot Z^{(n)}(o;z)$ converges for $|z| < R$ and it then follows (from the work of Bers and Gelbart) that $f(z) = \sum_0^\infty \frac{f^{[n]}(o)}{n!}\cdot Z^{(n)}(o;z)$ for $|z| < R$.

As the first step in this programme we have the following result:

THEOREM 2.1: Suppose $Z^{(n)}(z_o;z) = u^{(n)} + i\,v^{(n)}$

and $i\cdot Z^{(n)}(z_o;z) = \bar{u}^{(n)} + i\,\bar{v}^{(n)}$

where $u^{(n)}, v^{(n)}, \bar{u}^{(n)}, \bar{v}^{(n)}$ are real and n is an integer ≥ 0.

Then

$$u^{(2n)}(z_0;z) = (-1)^n (y-y_0)^{2n} + \frac{(-1)^n\, 2n(x-x_0)}{\pi} \int_0^1 e^{cu(x-x_0)} (1-u)^{1/2}\, u^{-1/2} \cdot$$

$$\left\{ \int_0^\pi i \cos\Theta \left[y-y_0 + 2i(x-x_0)[u(1-u)]^{1/2} \cos\Theta \right]^{2n-1} d\Theta \right\} du$$

$$= (-1)^n (y-y_0)^{2n} + (-1)^n\, 2(x-x_0)^2 \int_0^1 e^{cu(x-x_0)} (1-u)\, [(y-y_0)^2 +$$

$$4(x-x_0)^2\, u(1-u)]^{n-1} \left\{ 2n\, P_{2n}\left(\frac{y-y_0}{[(y-y_0)^2 + 4(x-x_0)^2\, u(1-u)]^{1/2}} \right) \right.$$

$$\left. - \frac{y-y_0}{[(y-y_0)^2 + 4(x-x_0)^2\, u(1-u)]^{1/2}}\, P'_{2n}\left(\frac{y-y_0}{[(y-y_0)^2 + 4(x-x_0)^2 u(1-u)]^{1/2}} \right) \right\} du$$

where $P_n(z)$ is Legendre's Polynomial.

$$u^{(2n+1)}(z_0;z) = \frac{(-1)^n e^{cx_0}\, (2n+1)(x-x_0)}{\pi} \int_0^1 e^{cu(x-x_0)} \left[\int_0^\pi (y-y_0 + 2i(x-x_0)[u(1-u)]^{1/2} \cdot \right.$$

$$\left. \cdot \cos\Theta)^{2n}\, d\Theta \right] du$$

$$= (-1)^n e^{cx_0} (2n+1)(x-x_0) \int_0^1 e^{cu(x-x_0)}\, [(y-y_0)^2 + 4(x-x_0)^2\, u(1-u)]^n \cdot$$

$$\cdot P_{2n}\left(\frac{y-y_0}{[(y-y_0)^2 + 4(x-x_0)^2 u(1-u)]^{1/2}} \right) du \;;$$

$$v^{(2n)}(z_0;z) = e^{-cx_0}\, \frac{2n(-1)^{n-1}\, (x-x_0)}{\pi} \int_0^1 e^{-cu(x-x_0)} \left[\int_0^\pi (y-y_0 + 2i(x-x_0) \cdot \right.$$

$$\left. [u(1-u)]^{1/2} \cos\Theta)^{2n-1}\, d\Theta \right] du$$

$$= e^{-cx_o}(-1)^{n-1}\, 2n(x-x_o)\int_0^1 e^{-cu(x-x_o)}\,[(y-y_o)^2+4(x-x_o)^2u(1-u)]^{n-1/2}\,\cdot$$

$$\cdot\, P_{2n-1}\left(\frac{y-y_o}{[(y-y_o)^2+4(x-x_o)^2u(1-u)]^{1/2}}\right)du\,.$$

$$v^{(2n+1)}(z_o;z) = (-1)^n(y-y_o)^{2n+1} + \frac{(-1)^n(x-x_o)(2n+1)}{\pi}\int_0^1 (1-u)^{1/2}\,u^{-1/2}\,\cdot$$

$$e^{-cu(x-x_o)}\left[\int_0^{\pi} i\cos\Theta\,\left[(y-y_o)+2i(x-x_o)[u(1-u)]^{1/2}\cos\Theta\right]^{2n} d\Theta\right]du$$

$$= (-1)^n(y-y_o)^{2n+1} + (-1)^n\, 2(x-x_o)^2\int_0^1 e^{-cu(x-x_o)}(1-u)\,\cdot$$

$$[(y-y_o)^2+4(x-x_o)^2u(1-u)]^{n-1/2}\left\{(2n+1)\,\cdot\right.$$

$$\cdot\, P_{2n+1}\left(\frac{y-y_o}{[(y-y_o)^2+4(x-x_o)^2\,u(1-u)]^{1/2}}\right) - \frac{y-y_o}{[(y-y_o)^2+4(x-x_o)^2u(1-u)]^{1/2}}\,\cdot$$

$$\left.\cdot\, P'_{2n+1}\left(\frac{y-y_o}{[(y-y_o)^2+4(x-x_o)^2\,u(1-u)]^{1/2}}\right)\right\}du$$

The expressions for $\bar{u}^{(2n)}(z_o;z)$, $\bar{u}^{(2n+1)}(z_o;z)$, $\bar{v}^{(2n)}(z_o;z)$, $\bar{v}^{(2n+1)}(z_o;z)$ become obvious from Theorem 1.2.

PROOF: We note that the second form involving $P_n(z)$ is easily obtained from the first by using Laplace's Integral

$$\frac{1}{\pi}\int_0^{\pi}[y-y_o+i\,\zeta\cos\phi]^n\,d\phi = [(y-y_o)^2+\zeta^2]^{n/2}\,P_n\left(\frac{y-y_o}{[(y-y_o)^2+\zeta^2]^{1/2}}\right).$$

The first forms for $u^{(n)}$, $v^{(n)}$ (and thus $\tilde{u}^{(n)}$, $\tilde{v}^{(n)}$) are obtained from Definition 1.1 and Theorem 1.2 by assuming that $Z^{(n)}$ is the function corresponding to z^n and that $i \cdot Z^{(n)}$ is the function corresponding to iz^n.

Thus Theorem 1.2 shows that these functions are Σ-monogenic in the whole plane.

Next, it is easily seen that for these functions

$$\frac{d_\Sigma}{d_\Sigma z} Z^{(n)}(z_o;z) = n \tilde{Z}^{(n-1)}(z_o;z)$$

$$\frac{d_{\Sigma'}}{d_{\Sigma'} z} \tilde{Z}^{(n)}(z_o;z) = n Z^{(n-1)}(z_o;z) \quad , \quad n \geq 1 .$$

Consider, for instance, the case $n = 2p$, n an even integer.

$$\frac{d_\Sigma}{d_\Sigma z} Z^{(2p)}(z_o;z) = \frac{\partial v^{(2p)}}{\partial y} - i \frac{\partial u^{(2p)}}{\partial y} \qquad \text{from } (1.2)$$

$$= \frac{2p(2p-1)\, e^{-cx_o} (-1)^{p-1}(x-x_o)}{\pi} \int_0^1 e^{-cu(x-x_o)} \cdot$$

$$[\int_0^\pi (y-y_o + 2i(x-x_o)[u(1-u)]^{1/2} \cos\Theta)^{2p-1} d\Theta]\, du$$

$$-i \left\{(-1)^p\, 2p(y-y_o)^{2p-1} + \frac{(-1)^p\, 2p(2p-1)(x-x_o)}{\pi}\right.$$

$$\int_0^1 e^{cu(x-x_o)} (1-u)^{1/2} u^{-1/2} \left\{\int_0^\pi i \cos\Theta \cdot\right.$$

$$\left.\left. \cdot [y-y_o + 2i(x-x_o)[u(1-u)]^{1/2} \cos\Theta]^{2p-2} d\Theta\right\} du\right\}$$

$$= 2p\, \tilde{Z}^{(2p-1)}(z_o;z)$$

since, clearly, $\tilde{Z}^{(n)}$ differs from $Z^{(n)}$ only by changing c to $-c$.

Next, it is obvious that

$$Z^{(n)}(z_o;z_o) = \tilde{Z}^{(n)}(z_o;z_o) = 0 \quad , \quad n \geq 1$$

$$i\cdot Z^{(n)}(z_o;z_o) = i\cdot\tilde{Z}^{(n)}(z_o;z_o) = 0 \quad , \quad n \geq 1 \, .$$

Thus, for $n \geq 1$,

$$Z^{(n)}(z_o;z) = n\int_{z_o}^{z} \tilde{Z}^{(n-1)}(z_o;\zeta)\, d_{\Sigma'}\zeta$$

$$i\cdot Z^{(n)}(z_o;z) = n\int_{z_o}^{z} i\cdot\tilde{Z}^{(n-1)}(z_o;\zeta)\, d_{\Sigma'}\zeta$$

$$\tilde{Z}^{(n)}(z_o;z) = n\int_{z_o}^{z} Z^{(n-1)}(z_o;\zeta)\, d_{\Sigma}\zeta$$

$$i\cdot\tilde{Z}^{(n)}(z_o;z) = n\int_{z_o}^{z} i\cdot Z^{(n-1)}(z_o;\zeta)\, d_{\Sigma}\zeta \, .$$

Also,

$$Z^{(o)}(z_o;z) = \tilde{Z}^{(o)}(z_o;z) = 1$$

$$i\cdot Z^{(o)}(z_o;z) = i\cdot\tilde{Z}^{(o)}(z_o;z) = i \, .$$

Thus the formulae we have given above for $Z^{(n)}(z_o;z)$ and $i\cdot Z^{(n)}(z_o;z)$ agree with those obtained by successive integration as in equations (1.4). Q.E.D.

COROLLARY 2.1: We can also define $L(z_o;z)$ and $i\cdot L(z_o;z)$ as the functions corresponding to $\log(z-z_o)$ and $i\log(z-z_o)$ respectively and $Z^{(\alpha)}(z_o;z)$, $i\cdot Z^{(\alpha)}(z_o;z)$ as the functions corresponding to $(z-z_o)^{\alpha}$, $i(z-z_o)^{\alpha}$ for α any real number. These definitions give Σ-monogenic functions for $y > y_o$ (or $y < y_o$).

Then in $y > y_o$ (or $y < y_o$)

$$\frac{d_\Sigma L(z_o;z)}{d_\Sigma z} = \mathcal{Z}^{(-1)}(z_o;z)$$

$$\frac{d_\Sigma i\cdot L(z_o;z)}{d_\Sigma z} = i\cdot\mathcal{Z}^{(-1)}(z_o;z)$$

$$\frac{d_\Sigma Z^{(\alpha)}(z_o;z)}{d_\Sigma z} = \alpha\,\mathcal{Z}^{(\alpha-1)}(z_o;z)$$

$$\frac{d_\Sigma i\cdot Z^{(\alpha)}(z_o;z)}{d_\Sigma z} = \alpha\, i\cdot\mathcal{Z}^{(\alpha-1)}(z_o;z) \quad .$$

The proof is the same as in the above theorem.

Next, bounds for $|Z^{(n)}(z_o;z)|$ and $|i\cdot Z^{(n)}(z_o;z)|$ are obtained. Without loss of generality, let $z_o = 0$.

LEMMA 2.1: For $z_o = 0$, and n an integer ≥ 0,

$$|u^{(2n)}| \leq e^{|cx|}\,(2n+1)\,r^{2n}$$

$$|u^{(2n+1)}| \leq e^{|cx|}\,(2n+1)\,r^{2n+1}$$

$$|v^{(2n)}| \leq e^{|cx|}\,(2n)\,r^{2n}$$

$$|v^{(2n+1)}| \leq e^{|cx|}\,(2n+2)\,r^{2n+1}$$

$$|\bar{u}^{(2n)}| \leq e^{|cx|}\,(2n)\,r^{2n}$$

$$|\bar{u}^{(2n+1)}| \leq e^{|cx|}\,(2n+2)\,r^{2n+1}$$

$$|\bar{v}^{(2n)}| \leq e^{|cx|}\,(2n+1)\,r^{2n}$$

$$|\bar{v}^{(2n+1)}| \leq e^{|cx|}\,(2n+1)\,r^{2n+1}$$

where $r = (x^2+y^2)^{1/2} = |z|$.

PROOF: We will show the calculations involved in obtaining the bounds on $u^{(2n)}(o;z)$, as the others are analogous.

$$\begin{aligned} \text{Since} \quad |y+2ix[u(1-u)]^{1/2} \cos\Theta|^2 &= y^2 + 4x^2u(1-u)\cos^2\Theta \\ &\leq y^2 + x^2 \qquad\qquad (0 \leq u \leq 1) \\ &= |x+iy|^2 , \end{aligned}$$

$$\begin{aligned} \text{then} \quad \left|\frac{1}{\pi}\int_0^\pi i\cos\Theta(y+2ix[u(1-u)]^{1/2}\cos\Theta)^{2n-1}d\Theta\right| &\leq \frac{1}{\pi}|x+iy|^{2n-1}\int_0^\pi |\cos\Theta|\, d\Theta \\ &= \frac{2}{\pi}|x+iy|^{2n-1} . \end{aligned}$$

Now from the first formula of Theorem 2.1 we have

$$u^{(2n)}(o;z) = (-1)^n \left\{ y^{2n} + \frac{2nx}{\pi}\int_0^1 e^{cux}(1-u)^{1/2}\, u^{-1/2}\, du \int_0^\pi i\cos\Theta \cdot \right.$$

$$\left. \cdot\, (y + 2ix\,[u(1-u)]^{1/2}\cos\Theta)^{2n-1}\, d\Theta \right\} .$$

Therefore,

$$\begin{aligned} |u^{(2n)}(o;z)| &\leq y^{2n} + \frac{4n|x|}{\pi} e^{|cx|}\,|x+iy|^{2n-1}\int_0^1 (1-u)^{1/2}\,u^{-1/2}\,du \\ &= y^{2n} + 2n|x|\,e^{|cx|}\,|x+iy|^{2n-1} \\ &\leq e^{|cx|}|x+iy|^{2n-1}\,[|y| + 2n|x|] . \end{aligned}$$

Clearly, $\max\{|y| + 2n|x|\}$ is $r(4n^2+1)^{1/2}$.

$$\begin{aligned} \text{Hence} \quad |u^{(2n)}| &\leq e^{|cx|}|x+iy|^{2n}\,(4n^2+1)^{1/2} \\ &\leq e^{|cx|}(2n+1)\, r^{2n} . \qquad\qquad \text{Q.E.D.} \end{aligned}$$

From this Lemma, the following theorem then follows.

THEOREM 2.2: $|a\cdot Z^{(n)}(o;z)| < 2^{3/2}(n+1)\ |a|\ e^{|cx|}\ r^n \quad , \quad n \geq 1 \ ,$

where $r = (x^2+y^2)^{1/2}$.

PROOF: Let $a = \alpha + i\beta$ where α and β are real.

$$\begin{aligned}
\text{Then,}\quad |a\cdot Z^{(2n)}(o;z)| &= |\alpha\, Z^{(2n)}(o;z) + \beta\, i\cdot Z^{(2n)}(o;z)| \\
&= |\alpha\, u^{(2n)} + i\,\beta\,\bar{v}^{(2n)} + i\,\alpha\, v^{(2n)} + \beta\,\bar{u}^{(2n)}| \\
&\leq 2^{1/2}|a|\ [|u^{(2n)}|^2 + |\bar{u}^{(2n)}|^2 + |v^{(2n)}|^2 + |\bar{v}^{(2n)}|^2]^{1/2}
\end{aligned}$$

(by Schwarz's inequality)

$$\begin{aligned}
&\leq 2|a|\ e^{|cx|}\ r^{2n}\ \{(2n+1)^2 + (2n)^2\}^{1/2} \\
&< 2^{3/2}\ (2n+1)\ |a|\ e^{|cx|}\ r^{2n}.
\end{aligned}$$

$$\begin{aligned}
\text{Also,}\quad |a\cdot Z^{(2n+1)}(o;z)| &= |\alpha\, u^{(2n+1)} + i\,\beta\,\bar{v}^{(2n+1)} + i\,\alpha\, v^{(2n+1)} + \beta\,\bar{u}^{(2n+1)}| \\
&\leq 2^{1/2}\ |a|\ [|u^{(2n+1)}|^2 + |\bar{v}^{(2n+1)}|^2 + |v^{(2n+1)}|^2 + \\
&\qquad + |\bar{u}^{(2n+1)}|^2]^{1/2} \\
&\leq 2^{1/2}\ |a|\ \{2(2n+1)^2 + 2(2n+2)^2\}^{1/2}\ e^{|cx|}\ r^{2n+1} \\
&< 2^{3/2}\ |a|\ (2n+2)\ e^{|cx|}\ r^{2n+1}.
\end{aligned}$$

Q.E.D.

COROLLARY 2.2: $|a\cdot Z^{(n)}(z_o;z)| < 2^{3/2}\ (n+1)\ |a|\ e^{|cx|+|cx_o|}\ r^n$

where $r = [(x-x_o)^2 + (y-y_o)^2]^{1/2}$ and $n \geq 0$.

PROOF: Everything is as with $a \cdot Z^{(n)}(o;z)$ except that x is replaced by $(x-x_o)$, y by $(y-y_o)$ and there is a factor e^{cx_o} or e^{-cx_o}. Now,

$$\begin{aligned} |e^{cx_o - cux_o}| &\leq e^{|cx_o|(1-u)} \quad , \qquad 0 \leq u \leq 1 \quad , \\ &\leq e^{|cx_o|} \end{aligned}$$

and so the result follows.

CHAPTER III

ESTIMATES ON THE DERIVATIVES OF $f(z)$ AND THE GENERALIZED TAYLOR'S THEOREM.

We wish to show that if $f(z)$ is Σ-monogenic in $|z| < R$, then

$$f(z) = \sum_{0}^{\infty} a_n \cdot Z^{(n)}(0;z)$$

where $a_n = \dfrac{f^{[n]}(0)}{n!}$.

Now we already have bounds on $|Z^{(n)}(0;z)|$ and so we shall now obtain bounds on $f^{[n]}(0)$.

Here,

$$f^{[0]}(z) = f(z) = u + iv$$

$$f^{[1]}(z) = \frac{d_\Sigma f}{d_\Sigma z} = v_y - iu_y$$

$$f^{[n]}(z) = \begin{cases} \dfrac{d_\Sigma f^{[n-1]}(z)}{d_\Sigma z}, & \text{if } n \text{ is odd} \\[2ex] \dfrac{d_{\Sigma'} f^{[n-1]}(z)}{d_{\Sigma'} z}, & \text{if } n \text{ is even}. \end{cases}$$

Hence,

$$\begin{aligned} f^{[2n]}(z) &= (-1)^n \frac{\partial^{2n} u}{\partial y^{2n}} + i(-1)^n \frac{\partial^{2n} v}{\partial y^{2n}} \\ f^{[2n-1]}(z) &= (-1)^{n-1} \frac{\partial^{2n-1} v}{\partial y^{2n-1}} + i(-1)^n \frac{\partial^{2n-1} u}{\partial y^{2n-1}}. \end{aligned} \qquad (3.1)$$

(n a positive integer)

$$\left.\begin{aligned} &\text{Let} \quad X = e^{-\frac{c}{2}x} u \,, \quad Y = e^{\frac{c}{2}x} v \,. \\ \text{Then,} \quad &\Delta X - \frac{c^2}{4} X = 0 \,, \quad \Delta Y - \frac{c^2}{4} Y = 0 \,. \end{aligned}\right\} \qquad (3.2)$$

Bounds on the derivatives of X and Y will be found from the explicit solution of the Dirichlet Problem for $\Delta X - \frac{c^2}{4} X = 0$ in $|z| < R$.

First, it is easily seen that $K_o(\frac{c}{2}r)$, $I_n(\frac{c}{2}r)\cos n\theta$, $I_n(\frac{c}{2}r)\sin n\theta$, $(n \geq 1)$ are solutions of $\Delta X - \frac{c^2}{4} X = 0$, where (r,θ) are polar coordinates in the z-plane with origin $z = 0$.

Therefore, $K_o(\frac{c}{2}[r^2 + R^2 - 2rR\cos(\theta - \phi)]^{1/2})$ is also a solution.

Here, $I_n(z)$, $K_n(z)$ are Bessel functions of pure imaginary argument [4, pp. 77-78].

$$K_o(\tfrac{c}{2}r) = -\log(\tfrac{c}{4}r) - \gamma + O(r\log r)\,, \quad \text{for } r \text{ small.}$$

Using these solutions we can obtain a Green's function.

LEMMA 3.1: Let $G(Re^{i\phi}, re^{i\theta})$ be the Green's function for $\Delta u - \frac{c^2}{4} u = 0$; that is a solution of the equation with respect to r and θ in $|z| < a$, which is 0 on $|z| = a$ and which has a singularity $-\log|PP'|$ for $P = re^{i\theta}$ near $P' = Re^{i\phi}$. Then

$$G(Re^{i\phi}, re^{i\theta}) = K_o(\tfrac{c}{2}[r^2 + R^2 - 2Rr\cos(\theta-\phi)]^{1/2})$$

$$- \frac{K_o(\frac{c}{2}a)}{I_o(\frac{c}{2}a)} I_o(\tfrac{c}{2}r)\, I_o(\tfrac{c}{2}R) \tag{3.3}$$

$$- 2\sum_1^\infty \frac{I_n(\frac{c}{2}r)}{I_n(\frac{c}{2}a)} K_n(\tfrac{c}{2}a)\, I_n(\tfrac{c}{2}R)\cos n(\theta-\phi)$$

where $R < a$.

PROOF: From the asymptotic properties of I_n and K_n it is clear that the above series converges for $R < a$, and it can be differentiated term by term any number of times.

So the above G is a solution of the equation and it obviously has a singularity $-\log|PP'|$.

Next,

$$G(Re^{i\phi}, ae^{i\theta}) = K_o(\tfrac{c}{2}[a^2 + R^2 - 2aR\cos(\theta-\phi)]^{1/2})$$

$$- K_o(\tfrac{c}{2}a)\ I_o(\tfrac{c}{2}R)$$

$$- 2\sum_1^\infty K_n(\tfrac{c}{2}a)\, I_n(\tfrac{c}{2}R)\cos n(\theta-\phi) .$$

But this is zero by [4, p. 361]. Q.E.D.

LEMMA 3.2: If $\Delta X - \frac{c^2}{4}X = 0$ for $R < a$ where $x = R\cos\phi$, $y = R\sin\phi$, and if X is $C^{(2)}$ for $R \leq a$, then

$$X(Re^{i\phi}) = \frac{1}{2\pi}\int_0^{2\pi} \left[\frac{I_o(\frac{c}{2} R)}{I_o(\frac{c}{2} a)} + 2\sum_1^{\infty} \frac{I_n(\frac{c}{2} R)}{I_n(\frac{c}{2} a)} \cos n\,(\Theta-\phi) \right] f(\Theta)d\Theta \tag{3.4}$$

for $R < a$, where $X(ae^{i\Theta}) = f(\Theta)$.

PROOF: In the usual manner, we can express $X(Re^{i\phi})$ by means of an integral over the boundary of $|z| = a$.

$$X(Re^{i\phi}) = -\frac{a}{2\pi}\int_0^{2\pi} X(ae^{i\Theta}) \left[\frac{\partial G(Re^{i\phi}, re^{i\Theta})}{\partial r} \right]_{r=a} d\Theta \quad .$$

Since

$$K_o'(x) = -K_1(x), \quad I_o'(x) = I_1(x),$$

$$-2K_n'(x) = K_{n-1}(x) + K_{n+1}(x),$$

$$2I_n'(x) = I_{n-1}(x) + I_{n+1}(x),$$

we have

$$\frac{\partial G(Re^{i\phi}, re^{i\Theta})}{\partial r} = -\frac{\frac{c}{2}(r-R\cos(\Theta-\phi))}{[R^2+r^2 - 2rR\cos(\Theta-\phi)]^{1/2}} K_1(\tfrac{c}{2}[R^2+r^2-2rR\cos(\Theta-\phi)]^{1/2})$$

$$-\frac{c}{2}\frac{K_o(\frac{c}{2} a)}{I_o(\frac{c}{2} a)} I_1(\tfrac{c}{2} r)\; I_0(\tfrac{c}{2} R)$$

$$-\frac{c}{2}\sum_1^{\infty}\frac{K_n(\frac{c}{2} a)\; I_n(\frac{c}{2} R)}{I_n(\frac{c}{2} a)} [I_{n-1}(\tfrac{c}{2} r) + I_{n+1}(\tfrac{c}{2} r)] \cos n(\Theta-\phi).$$

But, [4, p. 361]

$$\frac{a-R\cos(\theta-\phi)}{[a^2+R^2-2aR\cos(\theta-\phi)]^{1/2}}\ K_1\left(\frac{c}{2}[a^2+R^2-2aR\cos(\theta-\phi)]^{1/2}\right)$$

$$= K_1(\tfrac{c}{2}a)\ I_o(\tfrac{c}{2}R) + \sum_1^\infty I_m(\tfrac{c}{2}R)\ [K_{m+1}(\tfrac{c}{2}a) + K_{m-1}(\tfrac{c}{2}a)]\cos m(\theta-\phi)$$

and so

$$\left[\frac{\partial G(Re^{i\phi}, re^{i\theta})}{\partial r}\right]_{r=a} = -\frac{c}{2}\ \frac{I_o(\frac{c}{2}R)}{I_o(\frac{c}{2}a)}\ [K_o(\tfrac{c}{2}a)\ I_1(\tfrac{c}{2}a) + K_1(\tfrac{c}{2}a)\ I_o(\tfrac{c}{2}a)]$$

$$-\frac{c}{2}\sum_1^\infty \frac{I_n(\frac{c}{2}R)}{I_n(\frac{c}{2}a)}\ [K_n(\tfrac{c}{2}a)\ I_{n-1}(\tfrac{c}{2}a) + K_n(\tfrac{c}{2}a)\ I_{n+1}(\tfrac{c}{2}a)$$

$$+ I_n(\tfrac{c}{2}a)\ K_{n+1}(\tfrac{c}{2}a) + I_n(\tfrac{c}{2}a)\ K_{n-1}(\tfrac{c}{2}a)]\cos n(\theta-\phi)\ .$$

However, [4, p. 80]

$$I_n(x)\ K_{n+1}(x) + I_{n+1}(x)\ K_n(x) = \frac{1}{x}$$

and so

$$\left[\frac{\partial G(Re^{i\phi}, re^{i\theta})}{\partial r}\right]_{r=a} = -\frac{1}{a}\left\{\frac{I_o(\frac{c}{2}R)}{I_o(\frac{c}{2}a)} + 2\sum_1^\infty \frac{I_n(\frac{c}{2}R)}{I_n(\frac{c}{2}a)}\cos n(\theta-\phi)\right\}.$$

Substituting this expression into our initial expression for $X(Re^{i\phi})$, we get (3.4). Q.E.D.

We can obtain some interesting corollaries from (3.4) which we set down here even though they will not be used subsequently.

COROLLARY 3.1: The Mean Value Theorem.

$$X(R=0) = \frac{1}{2\pi \, I_o(\frac{c}{2} a)} \int_0^{2\pi} X(ae^{i\theta}) d\theta .$$

(Obvious).

COROLLARY 3.2: The Maximum Principle.

$$|X(R=0)| \leq \frac{\underset{\theta}{\text{Max}} \, |X(ae^{i\theta})|}{I_o(\frac{c}{2} a)} < \underset{\theta}{\text{Max}} \, |X(ae^{i\theta})| .$$

COROLLARY 3.3: If X is a solution in the whole plane of $\Delta u - \frac{c^2}{4} u = 0$, and

$$X = o\left(\frac{e^{\frac{c}{2} R}}{R^{1/2}}\right) \quad \text{for } R \text{ large},$$

then $X \equiv 0$.

COROLLARY 3.4: There are no entire solutions bounded at infinity of $\Delta u - \frac{c^2}{4} u = 0$ except $u \equiv 0$. In fact, there are no polynomial solutions except $u \equiv 0$.

Using equation (3.4) we can now obtain expressions for the derivatives of X. We first prove the following lemma.

LEMMA 3.3:

$$\left[\frac{\partial^p (I_n(\frac{c}{2} r) \cos n\Theta)}{\partial y^p}\right]_{x=y=o} = \begin{cases} \dfrac{(-1)^{\frac{n}{2}} \, p! \, (\frac{c}{4})^p}{(\frac{p-n}{2})! \, (\frac{p+n}{2})!} , & p \text{ even and } n \text{ even and } p \geq n > 0 . \\ 0 , & p \text{ odd or } n \text{ odd or } p < n . \end{cases}$$

$$\left[\frac{\partial^p (I_n(\frac{c}{2} r) \sin n\Theta)}{\partial y^p}\right]_{x=y=o} = \begin{cases} \dfrac{(-1)^{\frac{n-1}{2}} \, p! (\frac{c}{4})^p}{(\frac{p-n}{2})! \, (\frac{p+n}{2})!} , & p \text{ odd and } n \text{ odd and } p \geq n > 0 . \\ 0 , & p \text{ even or } n \text{ even or } p < n . \end{cases}$$

$$\left[\frac{\partial^p \, I_o(\frac{c}{2} r)}{\partial y^p}\right]_{x=y=o} = \begin{cases} 0 , & p \text{ odd} \\ (\frac{c}{4})^p \, \dfrac{p!}{[(\frac{p}{2})!]^2} , & p \text{ even} , \end{cases}$$

where $x = r \cos \Theta$, $y = r \sin \Theta$.

PROOF: Take the last relation first and use the result [4, pp. 360-361]

$$I_o(\tfrac{c}{2} r) = I_o(\tfrac{c}{2} x) \, I_o(\tfrac{c}{2} y) + 2 \sum_{1}^{\infty} (-1)^m \, I_m(\tfrac{c}{2} x) \, I_m(\tfrac{c}{2} y) \cos \frac{m\pi}{2} .$$

Then the result follows immediately since

$$I_m(x=o) = 0 , \quad (m \geq 1) \quad \text{and}$$

$$I_o(\tfrac{c}{2} y) = \sum_{0}^{\infty} \frac{(\frac{c}{4})^{2m} \, y^{2m}}{(m!)^2} .$$

The other results follow in the same manner using

$$I_n(\tfrac{c}{2} r) \cos n\Theta = I_n(\tfrac{c}{2} x)\, I_o(\tfrac{c}{2} y) + \sum_1^{\infty} (-1)^m \cos \tfrac{m\pi}{2}\, I_m(\tfrac{c}{2} y)\, [I_{m+n}(\tfrac{c}{2} x) + I_{m-n}(\tfrac{c}{2} x)] .$$

$$I_n(\tfrac{c}{2} r) \sin n\Theta = \sum_1^{\infty} (-1)^m \sin \tfrac{m\pi}{2}\, I_m(\tfrac{c}{2} y)\, [I_{m+n}(\tfrac{c}{2} x) - I_{m-n}(\tfrac{c}{2} x)] .$$

Q.E.D.

LEMMA 3.4: For $r \geq 1$,

$$\left| \left[\frac{\partial^{2r-1} X(Re^{i\phi})}{\partial y^{2r-1}} \right]_{R=o} \right| \leq \frac{4}{\pi} M(\tfrac{c}{4})^{2r-1} (2r-1)! \sum_1^{r} \frac{1}{(r-m)!(r+m-1)!\, I_{2m-1}(\frac{ca}{2})}$$

$$\left| \left[\frac{\partial^{2r} X(Re^{i\phi})}{\partial y^{2r}} \right]_{R=o} \right| \leq \frac{4}{\pi} M(\tfrac{c}{4})^{2r} (2r)! \sum_0^{r} \frac{1}{(r-m)!\,(r+m)!\, I_{2m}(\frac{ca}{2})}$$

where $M = \text{Maximum } |X(ae^{i\phi})|$.

As before we are assuming that $\Delta X - \frac{c^2}{4} X = 0$ in $R < a$ and $X \in C^{(2)}$ for $R \leq a$.

PROOF: i) Differentiating (3.4) and using Lemma 3.3 we get, for $r \geq 1$,

$$\left[\frac{\partial^{2r-1} X(Re^{i\phi})}{\partial y^{2r-1}} \right]_{R=o} = \frac{1}{\pi} \int_0^{2\pi} f(\Theta) \left\{ \sum_{\substack{n=1 \\ (n \text{ odd})}}^{2r-1} \frac{(-1)^{\frac{n-1}{2}} (2r-1)!\,(\frac{c}{4})^{2r-1}}{(\frac{2r-1-n}{2})!\; (\frac{2r+n-1}{2})!\; I_n(\frac{ca}{2})} \sin n\Theta \right\} d\Theta$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(\Theta)\left\{\sum_{m=1}^{r} \frac{(-1)^m\,(2r-1)!\,(\frac{c}{4})^{2r-1}}{(r-m)!\,(r+m-1)!\,I_{2m-1}(\frac{ca}{2})}\sin(2m-1)\Theta\right\} d\Theta$$

and so

$$\left|\left[\frac{\partial^{2r-1}X(Re^{i\emptyset})}{\partial y^{2r-1}}\right]_{R=o}\right| \leq \frac{(2r-1)!\,(\frac{c}{4})^{2r-1}}{\pi}\sum_{1}^{r}\frac{\int_0^{2\pi} M|\sin(2m-1)\Theta|\,d\Theta}{(r-m)!\,(m+r-1)!\,I_{2m-1}(\frac{ca}{2})}$$

$$= \frac{4}{\pi}M(2r-1)!\,(\frac{c}{4})^{2r-1}\sum_{1}^{r}\frac{1}{(r-m)!\,(m+r-1)!\,I_{2m-1}(\frac{ca}{2})}$$

where we have used $\int_0^{2\pi}|\sin(2m-1)\Theta|\,d\Theta = 4$.

Note that using Schwarz's inequality on $\int_0^{2\pi} f(\Theta)\sin(2m-1)\Theta\,d\Theta$ gives a worse result since $2^{1/2}\pi > 4$.

ii) Again, using equation (3.4) and Lemma 3.3, we get

$$\left|\left[\frac{\partial^{2r}X(Re^{i\emptyset})}{\partial y^{2r}}\right]_{R=o}\right| = \left|\frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{(\frac{c}{4})^{2r}\,(2r)!}{(r!)^2\,I_o(\frac{ca}{2})} + 2\sum_{\substack{n=2\\(n\text{ even})}}^{2r}\frac{(-1)^{\frac{n}{2}}(2r)!(\frac{c}{4})^{2r}}{(\frac{2r-n}{2})!(\frac{2r+n}{2})!}\,\frac{\cos n\Theta}{I_n(\frac{ca}{2})}\right\}\cdot f(\Theta)\,d\Theta\right| .$$

$$= \frac{(\frac{c}{4})^{2r}(2r)!}{2\pi}\left|\int_0^{2\pi}\left\{\frac{1}{(r!)^2 I_o(\frac{ca}{2})} + 2\sum_{1}^{r}\frac{(-1)^m\cos 2m\Theta}{(r-m)!\,(r+m)!\,I_{2m}(\frac{ca}{2})}\right\} f(\Theta)\,d\Theta\right|$$

$$\leq M\left(\frac{c}{4}\right)^{2r}(2r)!\left\{\frac{1}{(r!)^2\, I_o(\frac{c}{2}a)} + \frac{4}{\pi}\sum_{1}^{r}\frac{1}{(r-m)!\,(r+m)!\, I_{2m}(\frac{ca}{2})}\right\}$$

$$< \frac{4}{\pi} M\left(\frac{c}{4}\right)^{2r}(2r)! \sum_{0}^{r}\frac{1}{(r-m)!\,(r+m)!\, I_{2m}(\frac{ca}{2})} \quad .$$

Q.E.D.

We next obtain bounds on the series

$$\sum_{1}^{r}\frac{1}{(r-m)!\,(m+r-1)!\, I_{2m-1}(\frac{ca}{2})}$$

and $\displaystyle\sum_{0}^{r}\frac{1}{(r-m)!\,(r+m)!\, I_{2m}(\frac{ca}{2})}$.

LEMMA 3.5:

$$\left|\left[\frac{\partial^{2r-1} X(Re^{i\phi})}{\partial y^{2r-1}}\right]_{R=o}\right| \leq \frac{4}{\pi}\,\frac{M[(2r-1)!]}{a^{2r-1}}\,\left[\frac{4}{ac}\, I_1\left(\frac{ac}{2}\right)\right] .$$

$$\left|\left[\frac{\partial^{2r} X(Re^{i\phi})}{\partial y^{2r}}\right]_{R=o}\right| \leq \frac{4}{\pi}\,\frac{M(2r)!}{a^{2r}}\, I_o\left(\frac{ac}{2}\right) .$$

PROOF: i) Consider first $\displaystyle\sum_{1}^{r}\frac{1}{(r-m)!\,(m+r-1)!\, I_{2m-1}(\frac{ca}{2})}$

$$= \frac{1}{(2r-1)!\, I_{2r-1}(\frac{ac}{2})}\sum_{1}^{r}\frac{(2r-1)!\, I_{2r-1}(\frac{ac}{2})}{(r-m)!\,(m+r-1)!\, I_{2m-1}(\frac{ac}{2})} \quad .$$

Now from [4, p. 153], or alternatively from the power series for $I_n(x)$ for x real and $n > 0$,

$$\frac{I_{n+2}(x)}{I_n(x)} = 1 - \frac{2(n+1)}{2(n+1) + \frac{x^2}{2(n+2)+\dots}}$$

$$\leq \frac{x^2}{4(n+1)(n+2)} .$$

Therefore

$$\frac{I_{2r-1}(\frac{ac}{2})}{I_{2m-1}(\frac{ac}{2})} = \frac{I_{2r-1}(\frac{ac}{2})}{I_{2r-3}(\frac{ac}{2})} \cdot \frac{I_{2r-3}(\frac{ac}{2})}{I_{2r-5}(\frac{ac}{2})} \cdots\cdots \frac{I_{2m+1}(\frac{ac}{2})}{I_{2m-1}(\frac{ac}{2})}$$

$$\leq (\tfrac{ac}{4})^{2r-2m} \frac{(2m-1)!}{(2r-1)!}$$

and so $$\sum_{1}^{r} \frac{1}{(r-m)!(m+r-1)!\ I_{2m-1}(\frac{ac}{2})} \leq \frac{1}{(2r-1)!\ I_{2r-1}(\frac{ac}{2})} \sum_{1}^{r} \frac{(2m-1)!}{(r-m)!\ (m+r-1)!} \cdot (\tfrac{ac}{4})^{2r-2m}$$

$$= \frac{1}{(2r-1)!\ I_{2r-1}(\frac{ac}{2})} \left\{ 1 + \frac{(\frac{ac}{4})^2}{1!(2r-2)} + \frac{(\frac{ac}{4})^4}{2!(2r-3)(2r-4)} + \cdots + \frac{(\frac{ac}{4})^{2r-2}}{(r-1)!\ r!} \right\}$$

$$= \frac{1}{(2r-1)!\ I_{2r-1}(\frac{ac}{2})} \sum_{q=o}^{r-1} \frac{(\frac{ac}{4})^{2q}}{q!\ [(2r-q-1)\ \cdots\ (2r-2q)]} .$$

Now in the general term of the above series, $r-1 \geq q$ and so $2r - 2q \geq 2$.

$$\text{Therefore} \sum_{0}^{r-1} \frac{(\frac{ac}{4})^{2q}}{q!\,[(2r-q-1)\ldots.(2r-2q)]} \leq \sum_{0}^{r-1} \frac{(\frac{ac}{4})^{2q}}{q!\,(q+1)!}$$

$$\leq \sum_{0}^{\infty} \frac{(\frac{ac}{4})^{2q}}{q!\,(q+1)!}$$

$$= \frac{4}{ac} \sum_{0}^{\infty} \frac{(\frac{ac}{4})^{2q+1}}{q!\,(q+1)!}$$

$$= \frac{4}{ac}\, I_1(\frac{ac}{2}) \quad .$$

Thus we have

$$\sum_{1}^{r} \frac{1}{(r-m)!\,(m+r-1)!\,I_{2m-1}(\frac{ac}{2})} \leq \frac{4}{ac}\, \frac{I_1(\frac{ac}{2})}{(2r-1)!\,I_{2r-1}(\frac{ac}{2})}$$

$$\leq \frac{I_1(\frac{ac}{2})}{(\frac{ac}{4})^{2r}}$$

and so from Lemma 3.4,

$$\left| \left[\frac{\partial^{2r-1} X(Re^{i\phi})}{\partial y^{2r-1}} \right]_{R=o} \right| \leq \frac{4}{\pi} M(2r-1)!\, (\frac{c}{4})^{2r-1} \frac{4\, I_1(\frac{ac}{2})}{ac\,(2r-1)!\, I_{2r-1}(\frac{ac}{2})}$$

$$\leq \frac{4M\,(2r-1)!}{\pi\, a^{2r-1}} \frac{I_1(\frac{ac}{2})}{(\frac{ac}{4})} \quad .$$

ii) Now consider

$$\sum_{o}^{r} \frac{1}{(r-m)!(r+m)!\ I_{2m}(\frac{ca}{2})} = \frac{1}{(2r)!\ I_{2r}(\frac{ac}{2})} \sum_{o}^{r} \frac{(2r)!\ I_{2r}(\frac{ac}{2})}{(r-m)!\ (r+m)!\ I_{2m}(\frac{ca}{2})} .$$

Again, as before

$$\frac{(2r)!\ I_{2r}(\frac{ac}{2})}{I_{2m}(\frac{ac}{2})} \leq (\frac{ac}{4})^{2r-2m} (2m)!$$

and so

$$\sum_{o}^{r} \frac{1}{(r-m)!(r+m)!\ I_{2m}(\frac{ac}{2})} \leq \frac{1}{(2r)!\ I_{2r}(\frac{ac}{2})} \sum_{o}^{r} \frac{(\frac{ac}{4})^{2r-2m}\ (2m)!}{(r-m)!\ (r+m)!}$$

$$= \frac{1}{(2r)!\ I_{2r}(\frac{ac}{2})} \left\{ 1 + \frac{(\frac{ac}{4})^2}{1!\ (2r-1)} + \frac{(\frac{ac}{4})^4}{2!(2r-2)(2r-3)} + \dots + \frac{(\frac{ac}{4})^{2r}}{r!\ r!} \right\}$$

$$= \frac{1}{(2r)!\ I_{2r}(\frac{ac}{2})} \sum_{o}^{r} \frac{(\frac{ac}{4})^{2q}}{q!\ [(2r-q)\ \dots\ (2r-2q+1)]}$$

$$\leq \frac{1}{(2r)!\ I_{2r}(\frac{ac}{2})} \sum_{o}^{\infty} \frac{(\frac{ac}{4})^{2q}}{q!\ q!}$$

$$= \frac{1}{(2r)!\ I_{2r}(\frac{ac}{2})}\ I_o(\frac{ac}{2})$$

$$\leq \frac{I_o(\frac{ac}{2})}{(\frac{ac}{4})^{2r}} .$$

Q.E.D.

We can now formulate the following theorem on $f^{[n]}(0)$.

THEOREM 3.1: If $f(z)$ is Σ-monogenic in $|z| \leq a$, with

$$\Sigma = \begin{pmatrix} e^{-cx} & 1 \\ e^{-cx} & 1 \end{pmatrix} ,$$

then for $n \geq 1$,

$$|f^{[2n]}(0)| \leq \frac{4}{\pi} \frac{2^{1/2} M(f)}{a^{2n}} (2n)! \; e^{\left|\frac{ca}{2}\right|} I_o\left(\frac{ac}{2}\right)$$

$$|f^{[2n-1]}(0)| \leq \frac{4}{\pi} \frac{2^{1/2} M(f) (2n-1)! \; e^{\left|\frac{ca}{2}\right|}}{a^{2n-1}} \frac{4}{a c} I_1\left(\frac{ac}{2}\right) ,$$

where $M(f) = \underset{(|z|=a)}{\text{Maximum}} |f(z)|$.

Alternatively,

$$|f^{[n]}(0)| \leq \frac{4}{\pi} \frac{M(f)}{a^n} 2^{1/2} n! \; e^{\left|\frac{ca}{2}\right|} I_o\left(\frac{ac}{2}\right) . \tag{3.5}$$

PROOF: Using equations (3.1) and (3.2) and Lemma 3.5,

$$|f^{[2n]}(0)| \leq \left\{ \left(\left[\frac{\partial^{2n} u}{\partial y^{2n}} \right]_{z=o} \right)^2 + \left(\left[\frac{\partial^{2n} v}{\partial y^{2n}} \right]_{z=o} \right)^2 \right\}^{1/2}$$

$$= \left\{ \left(\left[\frac{\partial^{2n} X}{\partial y^{2n}} \right]_{z=o} \right)^2 + \left(\left[\frac{\partial^{2n} Y}{\partial y^{2n}} \right]_{z=o} \right)^2 \right\}^{1/2}$$

$$\leq \frac{4}{\pi} \frac{(2n)!}{a^{2n}} I_o\left(\frac{ac}{2}\right) \left\{ \operatorname*{Max}_{|z|=a} \left(e^{\frac{-cx}{2}} u\right)^2 + \operatorname*{Max}_{|z|=a} \left(e^{\frac{cx}{2}} v\right)^2 \right\}^{1/2}$$

$$\leq \frac{4}{\pi} \frac{(2n)!}{a^{2n}} I_o\left(\frac{ac}{2}\right) e^{\left|\frac{ca}{2}\right|} \left\{ [M(f)]^2 + [M(f)]^2 \right\}^{1/2}$$

$$= \frac{4}{\pi} 2^{1/2} \frac{(2n)!}{a^{2n}} M(f) \; e^{\left|\frac{ca}{2}\right|} I_o\left(\frac{ac}{2}\right) .$$

Similarly we get the bound on $|f^{[2n-1]}(0)|$.

In the final result (3.5) we have used the fact that

$\frac{2}{x} I_1(x) \leq I_o(x)$. Q.E.D.

COROLLARY 3.5: If $f(z)$ is Σ-monogenic in $|z-z_o| < a$, then

$$|f^{[n]}(z_o)| \leq \frac{4}{\pi} 2^{1/2} \frac{M(f)}{a^n} n! \; e^{\left|\frac{ca}{2}\right| + |cx_o|} I_o\left(\frac{ac}{2}\right) .$$

From the bounds obtained on $|a \cdot Z^{(n)}(o;z)|$ and $|f^{[n]}(0)|$ the Generalized Taylor Theorem follows easily.

THEOREM 3.2: Suppose $f(z)$ is Σ-monogenic in $|z| < a$, where $\Sigma = \begin{pmatrix} e^{-cx} & 1 \\ e^{-cx} & 1 \end{pmatrix}$. Then

$$f(z) = \sum_{o}^{\infty} \frac{f^{[n]}(0)}{n!} \cdot Z^{(n)}(o;z)$$

for all z such that $|z| < a$.

The above series converges absolutely for $|z| < a$ and uniformly in any closed bounded circle $|z| \leq a-\epsilon$, $(\epsilon > 0)$.

PROOF: The statements on convergence follow immediately from Theorem 2.2 and equation (3.5).

Thus $\sum_{0}^{\infty} \frac{f^{[n]}(0)}{n!} \cdot Z^{(n)}(o;z)$ is Σ-monogenic for $|z| < a$.

However, Bers and Gelbart have shown that

$$f(z) = \sum_{0}^{\infty} \frac{f^{[n]}(0)}{n!} Z^{(n)}(o;z)$$

for $|z| < \alpha < a$. Therefore $\left\{ f(z) - \sum_{0}^{\infty} \frac{f^{[n]}(0)}{n!} \cdot Z^{(n)}(o;z) \right\}$ is a function which is Σ-monogenic in $|z| < a$ and equal to zero in some neighbourhood of the origin. Therefore it is identically zero in $|z| < a$. Q.E.D.

COROLLARY 3.6: If $f(z)$ is Σ-monogenic in $|z-z_o| < a$, then

$$f(z) = \sum_{0}^{\infty} \frac{f^{[n]}(z_o)}{n!} \cdot Z^{(n)}(z_o;z).$$

It is clear of course that the bound given for $f^{[n]}(0)$ in Theorem 3.1 is not the best possible. The proof of Lemma 3.5 shows that the factor $I_o(\frac{ca}{2})$ can be replace by a factor which approaches one as n approaches infinity.

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APPENDIX

Theorem 1.2 can be extended to more general domains by a proof that merely uses integration by parts and not series expansions. We give the result here.

THEOREM A.1: Let $f(z)$ be analytic in a domain each of whose points can be joined to the y-axis by a segment of a line parallel to the x-axis, such segment being interior to the domain. Then in such a domain Γ, define

i) $$u_1 = f(y) + \frac{x}{\pi}\int_0^1 e^{cux}(1-u)^{1/2}u^{-1/2}\,du\int_0^\pi i\cos\phi\, f'(y+2ix[u(1-u)]^{1/2}\cos\phi)\,d\phi$$

$$v_1 = -\frac{x}{\pi}\int_0^1 e^{-cux}\,du\int_0^\pi f'(y+2ix[u(1-u)]^{1/2}\cos\phi)\,d\phi\ .$$

Then, $$e^{-cx}u_x = v_y$$

$$e^{-cx}u_y = -v_x \qquad \text{in } \Gamma\ .$$

ii) $$u_2 = \frac{x}{\pi}\int_0^1 e^{cux}\,du\int_0^\pi f'(y+2ix[u(1-u)]^{1/2}\cos\phi)\,d\phi$$

$$v_2 = f(y) + \frac{x}{\pi}\int_0^1 e^{-cux}(1-u)^{1/2}u^{-1/2}\,du\int_0^\pi i\cos\phi\, f'(y+2ix[u(1-u)]^{1/2}\cos\phi)\,d\phi\ .$$

Then, $$e^{-cx}u_x = v_y$$

$$e^{-cx}u_y = -v_x \qquad \text{in } \Gamma\ .$$

PROOF: We need only prove i) since, regarding u_1 and v_1 as functions of c,

$$u_2(c) = -v_1(-c)$$
$$v_2(c) = u_1(-c) .$$

Then, $$\frac{\partial u_1(c)}{\partial x} = e^{cx} \frac{\partial v_1(c)}{\partial y}$$

or $$\frac{\partial u_1(-c)}{\partial x} = e^{-cx} \frac{\partial v_1(-c)}{\partial y}$$

or $$\frac{\partial v_2(c)}{\partial x} = -e^{-cx} \frac{\partial u_2(c)}{\partial y}$$

or $$e^{-cx} \frac{\partial u_2(c)}{\partial x} = - \frac{\partial v_2(c)}{\partial y} .$$

Similarly, $$e^{-cx} \frac{\partial u_2}{\partial x} = \frac{\partial v_2}{\partial y} .$$

We now proceed to prove i). (Call $u_1 = u$, $v_1 = v$).

$$v_y = - \frac{x}{\pi}\int_0^1 e^{-cux}\, du \int_0^\pi f''(y+2ix[u(1-u)]^{1/2} \cos \phi)\, d\phi$$

$$u_x = \frac{1}{\pi}\int_0^1 e^{cux} (1-u)^{1/2} u^{-1/2}\, du \int_0^\pi i \cos \phi\ f'(y+2ix[u(1-u)]^{1/2} \cos \phi)\, d\phi$$

$$- \frac{2x}{\pi}\int_0^1 e^{cux}(1-u)\, du \int_0^\pi \cos^2\phi\ f''(y+2ix[u(1-u)]^{1/2} \cos \phi)\, d\phi$$

$$+ \frac{cx}{\pi}\int_0^1 e^{cux} (1-u)^{1/2} u^{1/2}\, du \int_0^\pi i \cos \phi\ f'(y+2ix[u(1-u)]^{1/2} \cos \phi)\, d\phi .$$

So, $$e^{-cx}u_x = \frac{1}{\pi}\int_0^1 e^{-cxt}t^{1/2}(1-t)^{-1/2}\,dt\int_0^\pi i\cos\phi\, f'(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d\phi$$

$$-\frac{2x}{\pi}\int_0^1 e^{-cxt}\,t\,dt\int_0^\pi \cos^2\phi\, f''(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d\phi$$

$$+\frac{cx}{\pi}\int_0^1 e^{-cxt}\,t^{1/2}(1-t)^{1/2}\,dt\int_0^\pi i\cos\phi\, f'(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d\phi$$

$$= F_1 + F_2 + F_3$$

where we have made the substitution $t = 1-u$.

Now,

$$F_1 = \frac{1}{\pi}\int_0^1 e^{-cxt}\,t^{1/2}(1-t)^{-1/2}\,dt\int_0^\pi f'(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d(i\sin\phi)$$

$$= -\frac{1}{\pi}\int_0^1 e^{-cxt}\,t^{1/2}(1-t)^{-1/2}\,dt\int_0^\pi i\sin\phi(-2ix[t(1-t)]^{1/2}\sin\phi)\cdot$$

$$f''(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d\phi$$

[Integration by parts]

$$= -\frac{2x}{\pi}\int_0^1 e^{-cxt}\,t\,dt\int_0^\pi \sin^2\phi\, f''(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d\phi$$

so that

$$F_1 + F_2 = -\frac{2x}{\pi}\int_0^1 e^{-cxt}\,t\,dt\int_0^\pi f''(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d\phi\ .$$

Next, integrating by parts in F_3 , we get

$$F_3 = -\frac{1}{\pi}\int_0^1 t^{1/2}(1-t)^{1/2}\left\{\int_0^{\pi} i\cos\phi\; f'(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d\phi\right\} d(e^{-cxt})$$

$$= \frac{1}{\pi}\int_0^1 e^{-cxt}dt\left\{\frac{1}{2}\,t^{-1/2}(1-t)^{-1/2}(1-2t)\int_0^{\pi} i\cos\phi\; f'(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d\phi\right.$$

$$\left. + \, t^{1/2}(1-t)^{1/2}\int_0^{\pi}(-x\cos^2\phi)\; t^{-1/2}(1-t)^{-1/2}(1-2t)f''(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d\phi\right\}$$

$$= \frac{1}{2\pi}\int_0^1 e^{-cxt}(1-2t)\; t^{-1/2}(1-t)^{-1/2}dt\int_0^{\pi} i\cos\phi\; f'(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d\phi$$

$$- \frac{x}{\pi}\int_0^1 (1-2t)\; e^{-cxt}\; dt\int_0^{\pi}\cos^2\phi\; f''(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d\phi$$

$$= -\frac{1}{2\pi}\int_0^1 e^{-cxt}(1-2t)\; t^{-1/2}(1-t)^{-1/2}dt\int_0^{\pi}(-i\sin\phi)(2ix[t(1-t)]^{1/2}\sin\phi)\,\cdot$$

$$\cdot\; f''(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d\phi$$

$$- \frac{x}{\pi}\int_0^1 (1-2t)\; e^{-cxt}\; dt\int_0^{\pi}\cos^2\phi\; f''(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d\phi$$

$$= -\frac{x}{\pi}\int_0^1 e^{-cxt}\;(1-2t)\;dt\int_0^{\pi} f''(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d\phi\;.$$

So $$F_1 + F_2 + F_3 = -\frac{x}{\pi}\int_0^1 e^{-cxt}\;dt\int_0^{\pi} f''(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d\phi\;.$$

That is, $$e^{-cx}\,u_x = v_y\;.$$

Now also,

$$u_y = f'(y) + \frac{x}{\pi}\int_0^1 e^{cux}(1-u)^{1/2}u^{-1/2}\,du\int_0^\pi i\cos\phi\; f''(y+2ix[u(1-u)]^{1/2}\cos\phi)\,d\phi$$

$$v_x = -\frac{1}{\pi}\int_0^1 e^{-cux}\,du\int_0^\pi f'(y+2ix[u(1-u)]^{1/2}\cos\phi)\,d\phi$$

$$-\frac{x}{\pi}\int_0^1 e^{-cux}\,du\int_0^\pi 2i[u(1-u)]^{1/2}\cos\phi\quad f''(y+2ix[u(1-u)]^{1/2}\cos\phi)\,d\phi$$

$$+\frac{cx}{\pi}\int_0^1 e^{-cux}\,u\,du\int_0^\pi f'(y+2ix[u(1-u)]^{1/2}\cos\phi)\,d\phi$$

so that

$$e^{cx}v_x = -\frac{2x}{\pi}\int_0^1 e^{cxt}\,t^{1/2}(1-t)^{1/2}dt\int_0^\pi i\cos\phi\quad f''(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d\phi$$

$$-\frac{1}{\pi}\int_0^1 e^{cxt}\,dt\int_0^\pi f'(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d\phi$$

$$+\frac{cx}{\pi}\int_0^1 e^{cxt}\,(1-t)\,dt\int_0^\pi f'(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d\phi$$

$$= \quad G_1 + G_2 + G_3\,.$$

Now

$$G_3 = \frac{1}{\pi}\int_0^1 (1-t)\,d\,(e^{cxt})\int_0^\pi f'(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d\phi$$

$$= \frac{1}{\pi}\Big[(1-t)\int_0^\pi f'(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d\phi\quad e^{cxt}\Big]_0^1$$

$$+\frac{1}{\pi}\int_0^1 e^{cxt}dt\int_0^\pi f'(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d\phi$$

$$-\frac{1}{\pi}\int_0^1 e^{cxt}\,(1-t)dt\int_0^\pi 2ix\cos\phi\;\frac{1}{2}t^{-1/2}(1-t)^{-1/2}(1-2t)\,f''(y+2ix[t(1-t)]^{1/2}\cos\phi)\,d\phi$$

$$= - \frac{1}{\pi}\int_0^{\pi} f'(y)\, d\phi + \frac{1}{\pi}\int_0^1 e^{cxt}\, dt \int_0^{\pi} f'(y+2ix[t(1-t)]^{1/2} \cos\phi)\, d\phi$$

$$- \frac{x}{\pi}\int_0^1 e^{cxt}(1-t)^{1/2} t^{-1/2}\, dt \int_0^{\pi} (1-2t)\; i \cos\phi\; f''(y+2ix[t(1-t)]^{1/2} \cos\phi)\, d\phi$$

so that

$$G_3 + G_2 = -f'(y) - \frac{x}{\pi}\int_0^1 e^{cxt}\,(1-2t)(1-t)^{1/2}\, t^{-1/2}\, dt \int_0^{\pi} i \cos\phi \cdot$$

$$\cdot\; f''(y+2ix[t(1-t)]^{1/2} \cos\phi)\, d\phi\, .$$

Therefore,

$$G_1 + G_2 + G_3 = -f'(y) - \frac{x}{\pi}\int_0^1 e^{cxt}\,(1-t)^{1/2} t^{-1/2} dt \int_0^{\pi} i \cos\phi \cdot$$

$$\cdot\; f''(y+2ix[t(1-t)]^{1/2} \cos\phi)\, d\phi$$

$$= -u_y\, .$$

Thus $e^{cx} v_x = -u_y$ or $e^{-cx} u_y = -v_x$.

The above procedure is valid wherever $f(y+2ix[u(1-u)]^{1/2} \cos\phi)$ is analytic $(0 \leq u \leq 1\, ,\; 0 \leq \phi \leq \pi)$. Since $\max\, [u(1-u)]^{1/2} = \frac{1}{2}$, the domain Γ will satisfy the condition. Q.E.D.

COROLLARY A.1: The formula given in Theorem 1.1 for the solutions of $\Delta u - cu_x = 0$ and $\Delta v + cv_x = 0$ are valid in Γ .